Mathematical Aspects of Electoral Systems

M. M. Konstantinov

University of Architecture, Civil Engineering and Geodesy, 1, Hr. Smirnenski Blvd., 1046 Sofia, Bulgaria

Abstract. In this paper we consider some mathematical aspects of electoral systems. Sometimes the results from elections seem paradoxical although they are mathematically correct. These cases are known as electoral paradoxes. A number of paradoxes of proportional and majoritarian electoral systems are considered.

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INTRODUCTION

The electoral system in any state is a major factor for its political, economical and social development. The electoral system in a broad sense is the set of legal acts and accepted practices that govern the electoral process. In a narrow sense the electoral system is the means which transforms votes cast into seats, or mandates.

The existing more than 300 electoral systems all over the world may, in general, be divided into three large groups: majoritarian, proportional and mixed. A special case is the German electoral system which sometimes is wrongly defined as mixed. In fact, the German system is proportional with half of the seats being personified by a simple majoritarian system.

In this paper we consider a description of a proportional electoral system and a number of electoral paradoxes. The latter are mathematically correct and sometimes practically possible results of elections which contradict our intuition, see also [1, 2, 3].

In what follows we denote by \( \mathbb{R} \) and \( \mathbb{Q} \) the sets of real and rational numbers, and by \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) and \( \mathbb{Q}_+ \) – the sets of positive integer, nonnegative integer and nonnegative rational numbers, respectively. If \( X \) is a set then \( X^n = X \times X \times \cdots \times X \) (\( n \) times).

For \( a = (a_1, a_2, \ldots, a_n) \), \( b = (b_1, b_2, \ldots, b_n) \) \( \in \mathbb{R}^n \) we write \( a \preceq b \) when \( a_k \leq b_k \), \( k = 1, 2, \ldots, n \). The 1–norm of \( a \) is denoted as \( \|a\| = |a_1| + |a_2| + \cdots + |a_n| \). For \( x \geq 0 \) the symbol \( [x] \in \mathbb{N}_0 \) denotes the integer part of \( x \).

We also use the following notations: \( n \) – the number of parties taking part in the elections and eligible to gain a seat; \( S \) – the total number of seats which are allocated to parties; \( C_k \) – the name of the \( k \)–th party; \( v_k \) – the number of votes cast for party \( C_k \); \( v = (v_1, v_2, \ldots, v_n) \) \( \in \mathbb{N}^n \) – the vector of votes; \( V = v_1 + v_2 + \cdots + v_n \) – the total number of valid votes; \( V' = v/V \) – the vector of relative vote distribution; \( s_k \) \( \in \mathbb{N}_0 \) – the number of seats allocated to party \( C_k \), where \( s_1 + s_2 + \cdots + s_n = S \); \( s = (s_1, s_2, \ldots, s_n) \) \( \in \mathbb{N}_0^n \) – the vector of seats; \( s'/S \) – the vector of relative seat distribution; \( \sigma_k = s_k/V = s_k'/\mathbb{Q}_+ \) – the proportional share of seats for party \( C_k \); \( k = 1, 2, \ldots, n \). We note that \( \|v\| = V \),
\[ \|s\| = S \text{ and } \|v^*\| = \|s^*\| = 1. \]

A second interpretation of the above quantities is as follows. Let the country be divided into \( n \) regions \( C_1, C_2, \ldots, C_n \) with population \( v_1, v_2, \ldots, v_n \), respectively. Then \( s_k \) is the number of seats preassigned to region \( C_k \) and \( S \) is the total number of seats in the parliament.

**PROPORTIONAL SYSTEMS**

In this section we give a mathematical description of a proportional system taking into account the possible effects of a barrier for parties eligible to take part in the distribution of seats.

Suppose that several parties receive votes in elections for a parliament. If there is a barrier of \( B > 1 \) votes then only parties with votes not less than \( B \) shall participate in the allocation of seats. In this statement of the problem it may happen that only one party shall pass the barrier taking all \( S \) seats. It is even possible that no party passes the barrier and there is no parliament elected! Fortunately, such cases are not known in the electoral practice. But a high barrier was a reason for only two parties to form a parliament.

**Example 1.** In the Turkish parliamentary elections in 2003 the 10–percent barrier had been overcome by only two parties.

Suppose that at least two parties pass the barrier. Then excluding parties which do not pass the barrier as well as independent candidates, we come to the following statement of the problem of proportional seat allocation.

There are \( n \geq 2 \) parties \( C_1, C_2, \ldots, C_n \) participating in the distribution of a total of \( S \in \mathbb{N} \) seats. The numbers of votes and seats for party \( C_k \) are \( v_k \in \mathbb{N} \) and \( s_k \in \mathbb{N}_0 \), respectively, where in case of a barrier of \( B \) votes it is fulfilled \( v_k \geq B \). Let \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n \) and \( s = (s_1, s_2, \ldots, s_n) \in \mathbb{N}_0^n \) be the vectors of party votes and party seats with \( V = v_1 + v_2 + \cdots + v_n \) being the total number of votes for parties eligible to gain a seat. We also have \( s_1 + s_2 + \cdots + s_n = S \). If there is a barrier \( B \) then \( n \leq \lceil V/B \rceil \).

Thus the initial data consists of the vector of votes \( v \) and the total number of seats \( S \), while the output is the vector of seats \( s \).

The *proportional system*, or the rule that transforms votes into seats, is described by the vector function \((v, S) \mapsto s\), denoted as

\[
f = (f_1, f_2, \ldots, f_n) : \mathbb{N}^{n+1} \to \mathbb{N}_0^n.
\]

Here the number \( s_k \) of seats allocated to party \( C_k \) is given by \( s_k = f_k(v, S) \), or \( s = f(v, S) \). Thus the function \( f \) must satisfy the relation

\[
\|f(v, S)\| = \sum_{k=1}^n f_k(v, S) = S.
\]

For a proportional system it is natural to assume that \( v_j > v_k \) implies \( s_j \geq s_k \), see condition (4) below. In contrast, for a majoritarian system the case \( v_j > v_k \) and \( s_j < s_k \) is possible.
The function $f$ may not be purely deterministic due to the restriction (2) and a stochastic element (a tie break) is necessary. Indeed, if $n = 2$, $S = 1$ and $v_1 = v_2$ then no deterministic rule may distribute one seat between two parties with equal votes. In this case one should use a stochastic mechanism to decide which party takes the seat.

The quantity $v_k/s_k$ is the price of one seat of party $C_k$. It shows approximately how many votes are necessary for one party seat. Since difficulties may occur when $s_k = 0$, we may use another price, namely $s_k/v_k$, which shows how many seats are gained by one party vote. Usually $s_k/v_k \ll 1$ although the case $s_k/v_k > 1$ is formally possible.

The function $f$ from (1) should have some desirable properties connected with proportionality and monotonicity.

A general requirement for a proportional system is that party seats must be approximately proportional to party votes. This requirement may be expressed as

$$\frac{s_k}{v_k} \simeq \frac{S}{V}, \quad k = 1, 2, \ldots, n,$$

or $Vs \simeq Sv$, or $v^* \simeq s^*$, where $s^* = s/S$, $v^* = v/V$. Another natural requirement is that

$$f(\lambda v, S) = f(v, S), \quad \lambda \in \mathbb{N}.$$

For each function $f$ and data $v, S$ we may define the deviation from proportionality as

$$d_f(v, S) = \|s^* - v^*\| = \left\|\frac{f(v, S)}{S} - \frac{v}{V}\right\|.$$

For a reasonably designed proportional system one should have $d_f(v, S) \ll 1$. Another measure for the deviation from proportionality is

$$\delta_f(v, S) = \sum_{k=1}^n \left|\frac{s_k^*}{v_k^*} - 1\right| > d_f(v, S).$$

There are several families of proportional methods that obey the property (3) but for some of them the deviation from (3) may be quite impressive. However, we shall consider only methods obeying the monotonicity condition

$$(v_j - v_k)(s_j - s_k) \geq 0, \quad j \neq k,$$

which is a major characteristic of any purely proportional electoral system. Inequalities (4) mean that a party with more votes should not obtain less seats than a party with less votes.

Another natural monotonicity condition which is assumed further on is that the function $f_k$ is not decreasing in $v_k$, $k = 1, 2, \ldots, n$, namely

$$(v_k - \hat{v}_k)(f_k(v, S) - f_k(\hat{v}, S)) \geq 0, \quad k = 1, 2, \ldots, n,$$

for $\|v - \hat{v}\| = |v_k - \hat{v}_k|.$

A list of five natural looking (but not always achievable) properties of a proportional electoral system is given below.

The first property corresponds to the expectation that if more seats $S + T$ ($T \in \mathbb{N}$) are allocated instead of the initial $S$ seats (with the same party votes) then a party may not decrease its seats.

P1 The function $f$ satisfies the condition $f(v, S) \preceq f(v, S + T), \quad T \in \mathbb{N}.$
Property **P1** means that if more seats are distributed with the same vote vector $v$ then no party may loose a seat. The violation of this property is known as “Alabama paradox” since it had been observed when the American state Alabama should loose a seat if the number of seats in the American Congress would have been increased by 1.

If this paradox occurs then some parties loose a total of $L \geq 1$ seats while the other parties win a total of $L + T$ seats. We shall say that the Alabama paradox is of order $N \in \mathbb{N}$ if it occurs for $T = N$ but does not occur for $T = 1, 2, \ldots, N - 1$. The classical Alabama paradox had been observed for $T = 1$ and is thus of first order.

The second property deals with the hypothesis that a new party takes part in the elections and gets seats with the old parties keeping their votes $v_1, v_2, \ldots, v_n$ and $S$ being the same. Then one may assume that an old party should not win additional seats. Let $v \in \mathbb{N}^n$ and $s \in \mathbb{N}_0^n$ be the old distributions of votes and seats, and let a new party $C_{n+1}$ with $v_{n+1} \in \mathbb{N}$ votes be added. Denote by $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n, \tilde{s}_{n+1}) \in \mathbb{N}_0^{n+1}$ the vector of seats corresponding to the vector of votes $\tilde{v} = (v_1, v_2, \ldots, v_n, v_{n+1}) \in \mathbb{N}^{n+1}$ and let $\tilde{s}_{n+1} \geq 1$. That an old party should not win additional seats when a new party gets seats, may be formulated as follows.

**P2** The inequalities $\tilde{s}_k \leq s_k$, $k = 1, 2, \ldots, n$, are fulfilled.

Property **P2** means that when a new party is included (keeping $S$ the same) then an old party may not increase its seats. The violation of Property **P2** is called “New party paradox”. If this paradox occurs then some of the old parties get additional $L \geq 1$ seats, while the rest old parties loose $L + \tilde{s}_{n+1}$ seats.

Property **P2** may be formulated in a dual form which is more likely to happen. Suppose that after the elections Party $C_n$ is banned and looses its $s_n \geq 1$ seats. Suppose also that parties $C_1, C_2, \ldots, C_{n-1}$ now receive all $S$ seats (instead of $S - s_n$ seats before the ban) according to their votes $v_1, v_2, \ldots, v_{n-1}$. Denote by $\hat{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{n-1}) \in \mathbb{N}^{n-1}$ the new vector of seats with $\hat{s}_1 + \hat{s}_2 + \cdots + \hat{s}_{n-1} = S$. Then one may expect that no party shall loose a seat, i.e., that $\hat{s}_k \geq s_k$, $k = 1, 2, \ldots, n - 1$. The violation of any of the last $n - 1$ inequalities may be called “Banned party paradox”. In fact, it is a particular case of the Alabama paradox. Thus the New (or Banned) party paradox may be considered as a special case of the Alabama paradox.

The Banned party paradox may also occur when Party $C_n$ is not banned but has only $M < s_n$ names in its party list. Then there are $s_n - M$ additional seats to be distributed among the first $n - 1$ parties and the Alabama paradox may take place with $T = s_n - M$.

The third property is concerned with the case when two electoral regions increase their population with certain increments. Then it is natural to assume that the region with a larger increment may not loose a seat. In terms of parliamentary elections the formulation of this property is as follows. Let two parties, say $C_1$ and $C_2$, increase their votes from $v_1$ to $v_1 + u_1$ and from $v_2$ to $v_2 + u_2$. Denote by $\bar{s} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$ the vector of seats corresponding to the vote vector $\bar{v} = v + u$, where $u = (u_1, u_2, 0, 0, \ldots, 0)$. Then the third property may be formulated in the following form.

**P3** If $u_1 > u_2 \geq 1$, then $\bar{s}_1 \geq s_1$ and $\bar{s}_2 \leq s_2$.

Property **P3** means that if the first party gets more additional votes in comparison with the second party ($u_1 > u_2$), then the first party should not loose a seat. The violation
of this property is known as the “Population paradox”. It had been observed in the American electoral practice with states instead of parties and population instead of votes.

Another property that seems natural is connected with the possibility to augment the votes and seats of two parties in order to form a coalition (or, dually, to split the votes and seats of a given party). Here it is natural to assume that the other parties shall keep their seats. In particular one may expect that the coalition shall obtain the sum of seats of its two parties. If this does not happen it may be considered as a paradox.

To describe this situation let \( n \geq 3 \) and let the vector of votes \((v_1, v_2, \ldots, v_n)\) produces the vector of seats \((s_1, s_2, \ldots, s_n)\). If parties \( C_{n-1} \) and \( C_n \) form a coalition (adding up their votes) then we may consider a new vote vector \( v = (v_1, v_2, \ldots, v_{n-2}, v_{n-1} + v_n) \in \mathbb{N}^{n-1} \). Denote by \( \mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{n-1}) \in \mathbb{N}^{n-1} \) the vector of seats corresponding to \( v \). Then the fourth property is formulated as follows.

**P4** The equalities

\[
s_k = s_k, \quad k = 1, 2, \ldots, n - 2,
\]

are fulfilled.

In this case \( \mathcal{S}_{n-1} = s_{n-1} + s_n \). The inverse may not be true: the last equality does not imply (6). Thus a weak form of Property **P4** is simply the validity of the equality \( \mathcal{S}_{n-1} = s_{n-1} + s_n \).

The violation of property **P4** is said to be a “Coalition paradox”.

In its dual form the Coalition paradox deals with the possibility to split the votes and seats of a given party as follows. Suppose that instead of the vote vector \((v_1, v_2, \ldots, v_n)\) we have a new vote vector \( v' = (v_1, v_2, \ldots, v_{n-1}, u_n, u_{n+1}) \in \mathbb{N}^{n+1} \), where \( u_n + u_{n+1} = v_n \). Thus parties \( C_1, C_2, \ldots, C_{n-1} \) keep their votes, while the \( v_n \) votes of party \( C_n \) are split in two parts \( u_n \) and \( u_{n+1} \). Let \( s'_1, s'_2, \ldots, s'_{n-1}, v'_{n+1} \) be the numbers of seats corresponding to the vote distribution \( v' \). Then one may expect that the conditions \( s'_k = s_k, \quad k = 1, 2, \ldots, n - 1 \), are fulfilled. When they are violated we have a dual form of the Coalition paradox.

It is worth mentioning that while the Coalition paradox (violation of Property **P4**) may lead to a “strange” change of seats of any party by at most 1, The Alabama paradox (violation of **P1** and **P2**) and the Population paradox (violation of **P3**) may occur in a more severe form with a party changing its seats with several units. In particular under the Population paradox a party may change its seats from very few to almost \( S/2 \)!

If we want to fulfill the proportionality property exactly then we obtain the fractional seats (or shares)

\[
\sigma_k := \frac{q_H}{q_H} = \frac{v_k}{V}, \quad k = 1, 2, \ldots, n,
\]

where \( q_H := V/S \) is the so called Hare quota. Hence another desirable property may be formulated as follows.

**P5** The inequalities

\[
[k] \leq s_k \leq [k] + 1, \quad k = 1, 2, \ldots, n,
\]

are fulfilled.

Property **P5** is known as the Hare quota rule. The violation of Property **P5** will be called “Hare quota paradox”.

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If \( q \left( \frac{V}{S + 1} < q \leq \frac{V}{S} \right) \) is another quota (e.g., the Droop quota) then we may use the shares \( v_k/q \) instead of \( \sigma_k, k = 1, 2, \ldots, n \), in the formulation of Property \( P_5 \).

Further on we consider two methods for proportional distribution of seats which satisfy condition (4).

The Largest Remainder Method with Hare quota, or LRM/H (known in Europe as Hare/Niemeyer method, and in USA as Hamilton method), does not have the Properties \( P_1, P_2, P_3 \) and \( P_4 \) but has Property \( P_5 \). Moreover, LRM/H has been designed in order to satisfy \( P_5 \).

In contrast, the Highest Averages Method, or HAM (known also as the D'Hondt method), has Properties \( P_1 \) and \( P_2 \) but does not have Properties \( P_3, P_4 \) and \( P_5 \).

**PARADOXES OF PROPORTIONAL SYSTEMS**

In the previous section we have considered five properties \( P_1 \)–\( P_5 \) that a proportional electoral system should eventually have. Unfortunately, an election system cannot have all of them. We recall that conditions (4) and (5) will always be presupposed.

Any violation of Properties \( P_1 \)–\( P_5 \) is referred to as a proportional system paradox. Of course, such a paradox is not a mathematical inconsistency. It rather means that the system behaves in a way which seems strange, or contradicts the intuition.

A relatively recent result of Balinski and Young (1982) shows that no system is free of paradoxes [1]. Using the above notations this result may be formulated as follows.

**Theorem 2.** (Balinski and Young, 1982) There is no a proportional electoral system that satisfies properties \( P_1 \) (or, equivalently, \( P_2 \)), \( P_3 \) and \( P_5 \).

Paradoxes due to other factors such as lack of an obligatory turnout and use of barriers are also possible in proportional electoral systems and are briefly discussed below.

**Paradoxes Due to Small Turnout and Barriers**

In most proportional systems the results are accepted as valid even if the voter turnout (number of votes cast divided to the number of all voters in the voting rolls) is small. If there are no restrictions on the turnout then the case \( V = 1 \) is theoretically possible (the conditions \( v_k \geq 1 \) \( (k = 1, 2, \ldots, n) \) and \( n \geq 2 \) assumed above imply \( V \geq n \geq 2 \); we ignore this fact for a moment). This means that

A single vote may elect the whole \( S \)-member parliament!

This, of course, has never happened in real parliamentary elections. But levels of 15–percent turnout have already been observed in elections for European Parliament. This is not a paradox in the above defined sense but may be politically dangerous.

In most proportional systems not every party is eligible to compete for a seat. Only parties passing the so called barrier may receive seats. The relative barrier \( \beta \in (0, 1) \) is usually measured in percents and is a part of all valid votes (4.00 or 5.56 percent in Bulgaria, 5.00 percent in Germany and Romania, 10.00 percent in Turkey). Some countries use the natural relative barrier \( \beta = 1/S \).
The **absolute barrier** \( B = \beta V \) is the minimum number of votes that is necessary for a party to compete for a seat. If the number \( \beta V \) is not an integer then the barrier \( B \) is the next integer larger than \( \beta V \), i.e., \( B = \lceil \beta V \rceil + 1 \). Only parties \( C_k \) with \( v_k \geq B \) take part in the distribution of seats.

**Example 3.** In the Bulgarian proportional system for parliamentary elections the relative barrier \( \beta \) is 0.04. Hence the absolute barrier is \( B = 0.04 \times V \) votes, or the next larger integer. If there are 5540837 valid votes then \( 0.04 \times 5540837 = 221633.48 \) and hence the absolute barrier is \( B = 221634 \) votes. This was the case in the Bulgarian parliamentary elections in 1991 when only three parties won seats.

Until the end of this subsection we shall discuss the Bulgarian parliamentary elections with a relative barrier 0.04 and a total number of seats \( S = 240 \). Since \( 0.04 \times 240 = 9.6 \) it is clear that a party \( C_k \) either obtains no seats (if \( v_k < B \)) or it obtains 9 or more seats (if \( v_k \geq B \)). In fact, the least number of seats won in Bulgaria in five elections since 1991 had been 12.

In contrast to the previous section, here we do not exclude the case when one or more parties have votes below the barrier.

In what follows we shall say that “a single vote changes \( K \geq 1 \) seats” if for \( v \in \mathbb{N}^n \) there is \( \hat{v} \in \mathbb{N}^n \) such that \( \| v - \hat{v} \| = 1 \) and \( \| s - \hat{s} \| = K \), where \( \hat{s} \) is the vector of seats corresponding to \( \hat{v} \). Note that \( \hat{v} \neq v \) is among the vectors which are nearest to \( v \) relative to the 1-norm.

We first note that although the average price of one seat is \( V/S \gg 1 \), **one vote may change one or more seats**. This contradicts our intuition which expects that an average of \( V/(2S) \) votes should change a seat. A popular way to show this is as follows. Let the vector of votes \( v \) allocates \( s_k \geq 1 \) seats to party \( C_k \) with \( v_k \) votes. Reduce the votes of \( C_k \) by 1 resulting in a vector of votes \( \hat{v} \) which differs from \( v \) only in its \( k \)-th element \( \hat{v}_k = v_k - 1 \). If \( C_k \) still has \( s_k \) seats, repeat the procedure. At a certain step party \( C_k \) will lose at least one seat since otherwise it shall have seats with zero votes.

Less known is the following fact.

*A single vote may change 9 or more seats.*

**Example 4.** Suppose that party \( C_1 \) has \( v_1 = B \) votes. Then it has \( s_1 \geq 9 \) seats (most probably \( s_1 \geq 10 \)). If we reduce \( v_1 \) by 1 then party \( C_1 \) will fall below the barrier and shall receive no seats.

But there is more to come.

*A single vote may change all 240 seats!*

**Example 5.** Let \( n = 26 \) and \( V = 25m \), where \( 25 \leq m \in \mathbb{N} \). Then the barrier is \( B = 0.04 \times V = m \). Suppose also that \( v_1 = m \), \( v_2 = v_3 = \cdots = v_{25} = m - 1 \) and \( v_{26} = 24 \). Then only the first party passes the barrier and hence \( s_1 = 240 \), \( s_2 = s_3 = \cdots = s_{26} = 0 \). If we reduce \( v_1 \) by 1 the total numbers of votes becomes \( \hat{V} = 25m - 1 \). Next we have \( 0.04 \times V = m - 0.04 \) and the new barrier \( \hat{B} \) remains equal to \( m \). But now all parties are below the barrier and there is no parliament elected. In this case \( \| v - \hat{v} \| = 1 \) and \( \| s - \hat{s} \| = 240 \).
Of course, the story may be told in a reverse order. Let 25 parties have \( m - 1 \) votes each, the 26-th has 24 votes. Then \( V = 25m - 1 \), the barrier is \( B = m \) (the least integer larger than \( 0.04 \times V = m - 0.04 \)), \( s_1 = s_2 = \cdots = s_{26} = 0 \) and there is no parliament elected. But if a party with \( m - 1 \) votes somehow gets one vote more, it passes the barrier and receives all 240 seats.

Paradoxes of HAM

The Highest Averages Method, or HAM, finds a seat distribution that maximizes the quantity \( \min\{v_k/s_k : k = 1, 2, \ldots, n\} \). The algorithm realizing the HAM is as follows.

1. Order the \( n \) parties by the rule \( v_1 \geq v_2 \geq \cdots \geq v_n \) (if there are parties with equal votes the ordering is by ties).
2. Construct the \( S \times n \) matrix with elements \( h_{i,j} = v_j/i, i = 1, 2, \ldots, S, j = 1, 2, \ldots, n \).
3. Mark the first \( S \) greatest elements \( h_{i,j} \) in descending order of magnitude starting with \( h_{1,1} \). If there are equal elements \( h_{i,j} = h_{k,l} \) for \( |i - k| + |j - l| \geq 1 \) then first comes the element \( h_{i,j} \) when \( i < k \), or when \( i = k \) and \( j < l \).
4. Determine \( s_k, k = 1, 2, \ldots, n \), as the number of marked elements \( h_{i,j} \) with \( k = j \). All \( S \) seats are allocated.

More generally, a HAM type method is as follows. Let \( 0 < q_1 < q_2 < \cdots < q_S \) be an increasing sequence. Then we may define the quantities \( h_{i,j} \) from \( h_{i,j} = v_j/q_i \). In particular, for \( q_i = i \) we have the D'Hondt method, for \( q_i = 2i - 1 \) the Saint Laguë method and for \( q_i = 2i \) the Imperiali method.

It is easy to show that the following statement is valid.

Theorem 6. The HAM satisfies Properties P1, P2 and does not satisfy Properties P3, P4 and P5.

The HAM satisfies P1 and P2 as a direct corollary from the above definition. That HAM does not satisfy Properties P3, P4 and P5 may be shown by examples, see also Theorem 7 below.

The consequences of Theorem 6 may be dramatic as the next theorem suggests. To demonstrate this we shall compare the results produced by HAM and LRM/H under the conditions of the Bulgarian electoral system.

Theorem 7. In the Bulgarian electoral system with a total of \( S = 240 \) seats and a relative barrier \( \beta = 0.04 \) the difference between the seats allocated to the largest party by HAM and LRM/H may reach 6.

The proof is based on a direct computation of seats for \( n = 11, v_1 = 15v_2 \) and \( v_2 = v_3 = \cdots = v_{11} \).

Of course, Property P5 may be violated in HAM not only for the largest party but for some other party. The HAM has been used in Bulgarian parliamentary elections in 1991, 1994, 1997, 2001 and 2005. Here Property P5 has been violated three times: in 1994 and 1997 the first party got one seat more and in 2005 the second party took one seat more than Property P5 suggests.
In 2007 the seats for the Elections of members of the European Parliament from Bulgaria have been allocated by LRM/H. In this case HAM gives the same result.

**Paradoxes of LRM/H**

First we recall how LRM/H works for parties passing the barrier (if any).

1. The Hare quota $q_H = V/S$ is defined. To each party $C_k$ the proportional share (fractional number of seats) $\sigma_k = v_k/q_H$ is allocated. If $\sigma_k \in \mathbb{N}$ set $s_k = \sigma_k$.

2. If $\sigma_k \notin \mathbb{N}$ for some $k$ then party $C_k$ gets $[\sigma_k]$ “automatic” seats. Define the remainder $r_k = \sigma_k - [\sigma_k]$ and let $R \in \mathbb{N}$ be the sum of all remainders (we have $R < n$). Then the sum of automatic seats is $S - R$ and there are $R$ seats for further allocation.

3. Order the parties so as $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$ with ties if equal remainders occur.

4. Set $s_k = [\sigma_k] + 1$ for $k = 1, 2, \ldots, R$ and $s_k = [\sigma_k]$ for $k = R + 1, R + 2, \ldots, n$. All $S$ seats are allocated.

Another LRM type method with quota $q_{N-B} = V/(S + 1)$ is the so called Hagenbach–Bischoff method. However, when $v_k/(S + 1) \in \mathbb{N}$, $k = 1, 2, \ldots, n$, this method will produce $S + 1$ seats and one seat must be taken away by ties. A popular LRM method uses the Droop quota $q_D = [V/(S + 1)] + 1$. A simple quota that always produces exactly $S$ seats may be chosen as $q_\lambda = V/(S + \lambda)$, where $\lambda \in [0, 1)$. For $\lambda = 0$ we have $q_0 = q_H$.

As mentioned above, LRM/H is designed so as to satisfy property P5. So possible paradoxes in using LRM/H are in violation of Properties P1 and P2 – the Alabama paradox, P3 – the Population paradox and P4 – the Coalition paradox. The counterpart of Theorem 6 here is as follows.

**Theorem 8.** The LMR/H satisfies Property P5 and does not satisfy Properties P1 – P4.

Examples of Paradoxes P1 and P3 are known from the electoral practice of USA in XIX century in the framework of the second interpretation described in the Introduction ($C_k$ are interpreted as states of USA). These examples are by no means minimal with regard to the number of parties (states), votes (population) and seats. Further examples with fewer parties are also known. Thus it is interesting to find examples of paradoxes with minimum number $n$ of parties and with minimum total number $S$ of seats (we recall that the inequality $n \geq 2$ is assumed in order to avoid trivial results). What a “minimal example” means should become clear from the next result.

**Theorem 9.** (i) If the Alabama paradox of first order occurs then $n \geq 3$ and $S \geq 3$. If the Alabama paradox of $N$–th order occurs then $n \geq N + 2$

(ii) The Alabama paradox of first order may occur for $n = 3$ and $S = 3$. The Alabama paradox of $N$–th order may occur for $n = N + 2$.

The proof for the first order paradox consists in two steps. First we show that if $n = 2$ or $S = 2$ then an Alabama paradox of first order may not occur. Next we find an example of a first order Alabama paradox with $n = S = 3$ which completes the proof. Such examples are described below.
Example 10. Let \( v_1 = v_2 = 13,000, v_3 = 4,000 \) and \( S = 3 \). Then the proportional shares are \( \sigma_1 = \sigma_2 = 1.3 \) and \( \sigma_3 = 0.4 \). Hence each party takes one seat, \( s_1 = s_2 = s_3 = 1 \). Suppose now that \( S = 4 \) (keeping the votes cast). Then the new proportional shares are \( \hat{\sigma}_1 = \hat{\sigma}_2 = 1.7(3) \) and \( \hat{\sigma}_3 = 0.5(3) \). Now parties \( C_1 \) and \( C_2 \) gain an additional seat \( (\hat{s}_1 = \hat{s}_2 = 2) \) while party \( C_3 \) looses its previous seat \( (\hat{s}_3 = 0) \).

Of course, this is not the only minimal example of an Alabama paradox. The set of all minimal Alabama paradoxes of first order is described below.

Consider elections with \( n = S = 3 \) and

\[
v_1 = \lambda (1 + x_1), \ v_2 = \lambda (1 + x_2), \ v_3 = \lambda (1 - x_1 - x_2),
\]

where \( \lambda = V/3 \) and the quantities \( x_1, x_2 \) satisfy \( x_1, x_2 > 0, x_1 + x_2 < 1 \). If

\[
2x_1 + x_2 < 1, \ x_1 + 2x_2 < 1 \tag{9}
\]

then the remainders \( r_1 = x_1 \) and \( r_2 = x_2 \) of parties \( C_1 \) and \( C_2 \) are less than the remainder \( r_3 = 1 - x_1 - x_2 \) of party \( C_3 \). Therefore \( s_1 = s_2 = s_3 = 1 \).

Suppose now that \( S = 4 \) with the same vote distribution. If \( x_1, x_2 < 1/2 \) and \( x_1 + x_2 > 1/4 \) then the remainders of the parties \( C_1, C_2 \) and \( C_3 \) will be \( \hat{r}_1 = (1 + 4x_1)/3 \), \( \hat{r}_2 = (1 + 4x_2)/3 \) and \( \hat{r}_3 = 4(1 - x_1 - x_2)/3 \). Moreover, if

\[
8x_1 + 4x_2 > 3, \ 4x_1 + 8x_2 > 3 \tag{10}
\]

then parties \( C_1 \) and \( C_2 \) will get two seats each, while party \( C_3 \) will have no seats.

Thus we have proved the following result.

**Theorem 11.** The minimal first order Alabama paradox \( n = S = 3 \) is described by the vote distribution (8), where the parameters \( x_1, x_2 \) satisfy the inequalities (9) and (10).

We shall not give an example of the New party paradox since it is a variant of the Alabama paradox.

Both LRM/H and HAM do not have Property P3. In case of two parties the vote vector is \((v_1, v_2)\) and the new vector is \(v = (v_1 + u_1, v_2 + u_2)\), where \( u_1 > u_2 \geq 1 \). If the Population paradox occurs then it is fulfilled \( \Delta = s_1 - \hat{s}_1 = v_1S/(v_1 + v_2) - (v_1 + u_1)S/(v_1 + v_2 + u_1 + u_2) = (v_1u_2 - v_2u_1)/((v_1 + v_2)(v_1 + v_2 + u_1 + u_2)) > 0 \) and hence \( u_1/v_1 < u_2/v_2 \).

Let us try to maximize \( \Delta \) as a function of \( v_k, u_k \). Since \( \Delta \) increases in \( v_2 \) and \( u_1 \) we may choose \( v_2 = 1 \) and \( u_1 = u_2 + 1 \). At the same time \( \Delta \) increases in \( u_2 \) and for \( u_2 \to \infty \) tends to \((1/2 - 1/(v_1 + 1))S\). Thus when \( u_2 \) and \( v_1 \) are large enough then \( s_1 \) decreases almost twice: from almost \( S \) to about \( S/2 \). Here LRM/H and HAM produce very close or identical results. In particular, both methods do not satisfy Property P3.

Consider now the Coalition paradox when LRM/H is applied. We shall discuss two minimal cases. In the first case the coalition looses a seat, while in the second case it wins a seat.

Let \( n = 3, S = 1 \) and

\[
v_1 = Vr_1, \ v_2 = Vr_2, \ v_3 = V(1 - r_1 - r_2), \ r_1 + r_2 < 1, \tag{11}
\]
where \( r_1, r_2 \in [0, 1] \). If \( r_1 > r_2 \) and \( r_1 > 1 - r_1 - r_2 \) then \( s_1 = 1 \) and \( s_2 = s_3 = 0 \). If Parties \( C_2 \) and \( C_3 \) make a coalition we shall have the vote distribution \( \tilde{v}_1 = V r_1, \tilde{v}_2 = V (1 - r_1) \).

Hence if \( r_1 < 1 - r_1 \) the new seats will be \( \tilde{s}_1 = 0, \tilde{s}_2 = 1 \).

Hence we have proved the following result.

**Theorem 12.** The minimal Coalition paradox, where the coalition gets more seats than the sum of seats of its parties, is described by the vote distribution (11) under the conditions \( 1/3 < r_1 < 1/2, 1 - 2r_1 < r_2 < r_1 \).

The HAM gives the same result for this case.

Let now \( n = 3, S = 2 \) and

\[
v_1 = V r_1, \; v_2 = V r_2, \; v_3 = V (2 - r_1 - r_2), \; r_1 + r_2 < 2,
\]

where \( r_1, r_2 \in [0, 1] \). If \( r_1 + r_2 > 1, r_1 < r_2 \) and \( r_1 < 2 - r_1 - r_2 \) then the seats will be \( s_1 = 0 \) and \( s_2 = s_3 = 1 \). When the second and third parties form a coalition we shall have \( \tilde{v}_1 = V r_1, \tilde{v}_2 = V (2 - r_1) \). When \( r_1 > 1/2 \) the numbers of seats will be \( \tilde{s}_1 = 1 \) and \( \tilde{s}_2 = 1 \).

Hence the coalition gets one seat less than the sum of seats of its parties. As a result we have the following result.

**Theorem 13.** The minimal Coalition paradox, where the coalition gets less seats than the sum of seats of its parties, is described by the vote distribution (12) under the conditions \( 1/2 < r_1 < 2/3, r_1 < r_2 < 2(1 - r_1) \).

**PARADOXES OF MAJORITARIAN SYSTEMS**

Paradoxes of majoritarian systems are well known. Moreover, unlike paradoxes of proportional systems which rarely happen, paradoxes of majoritarian systems have been observed many times in history. These paradoxes are one of the reasons which forced many democracies to abandon majoritarian systems and to start using more proportional electoral systems. This process may be observed even in the United Kingdom – the birthplace of the classical majoritarian system.

Consider a standard majoritarian system with relative or absolute majority (in one or two rounds, respectively). Then there are \( S > 1 \) one-seat electoral regions, or constituencies, and the candidate with the majority of votes in each region is considered elected.

A serious drawback of majoritarian systems is that due to geographical reasons or even to political manipulations the population in different constituencies may vary significantly. These differences may reach up to four times as in England. However, we shall not deal with such irregularities assuming that in all constituencies the number of votes cast is the same. We suppose also that this number is odd and denote it by \( 2q + 1, q \in \mathbb{N}_0 \). Even under the above assumption there are still several paradoxes of majoritarian systems. We discuss two such paradoxes denoting them by M1 and M2.

For simplicity we assume that only two parties \( C_1 \) and \( C_2 \) take part in the elections. We recall that \( v_k \) and \( s_k \) are the numbers of votes and seats for party \( C_k \), respectively.

**M1** Party \( C_1 \) with only \( S \) votes more than party \( C_2 \) \((v_1 = v_2 + S)\) takes all \( S \) seats.
In this case the first party receives one vote more than the second party in each of the $S$ constituencies.

An extreme form of this paradox is that Party $C_1$ with only $v_1 = S$ votes takes all $S$ seats. Here party $C_1$ gets one vote in each constituency while Party $C_2$ receives none. Another formulation of this paradox is the next statement.

$Party C_1$ with slightly more than $1/2$ of the votes takes all $S$ seats, while $Party C_2$ with slightly less than $1/2$ of the votes receives no seats.

**Example 14.** Suppose that in each constituency Party $C_1$ receives $q + 1$ votes while Party $C_2$ receives $q$ votes. Then $C_1$ wins all seats ($s_1 = S$) with a total of $v_1 = S(q + 1)$ votes, while Party $C_2$ gets no seats ($s_2 = 0$) with $v_2 = Sq$ votes. Moreover, we have $v_1/v_2 = 1 + 1/q \approx 1$.

This paradox has not happened in such a severe form. However, several times in Great Britain the Liberal Party had received only few seats with 25–30 percent of the votes. Of similar nature are the five cases in the political history of USA (the last one in 2000) when the candidate with less national votes won the presidency.

**M2** $Party C_1$ with slightly more than $1/4$ of the votes wins more than $1/2$ of the seats, while $Party C_2$ with slightly less than $3/4$ of the votes takes less than $1/2$ of the seats.

**Example 15.** Suppose that $S$ is odd. The total number of votes is $V = v_1 + v_2 = S(2q + 1)$. Let Party $C_1$ wins in $(S + 1)/2$ constituencies with $q + 1$ votes against $q$ votes for Party $C_2$. In the rest $(S - 1)/2$ constituencies Party $C_1$ has zero votes against $2q + 1$ votes for party $C_2$. Then $s_1 = (S + 1)/2 > s_2 = (S - 1)/2$, while $v_1 = V(1/4 + \xi)$, $v_2 = V(3/4 - \xi)$, where $\xi = (1/S + (S + 1)/V)/4 \ll 1$. Here the ratio of votes is about three times the ratio of seats.

Large deviations from proportionality are well known in countries using majoritarian systems. For example in France a party with 40% of the votes may win about 80% of the seats. In Great Britain and New Zealand a party with more than half of the votes takes less than half of the seats, etc.

**REFERENCES**